Family Name:__________________**Given Name:**__________________ **I.D.#**_________

MAT3320 Assignment 3

Total: 10 marks. Due date: Tuesday, June 27, on or before 4:00pm.

In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **on or before 4:00pm** on the due date. Late assignments will not be accepted.

1. (3 marks) Let $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x < 1; \\ 3 - x, & 1 \le x \le 2. \end{array} \right\}$. The Fourier sine series of $f(x)$ is

$$FSS(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right).$$

(i) (2 marks) Find b_6.

(ii) (1 mark) Find $FSS(2015)$.

Solution: (i).

$$\begin{aligned} b_6 &= \frac{2}{2}\int_0^2 f(x)\sin\frac{6\pi x}{2}dx = \int_0^1 \sin 3\pi x dx + \int_1^2 (3-x)\sin 3\pi x dx \\ &= \left[-\frac{1}{3\pi}\cos 3\pi x\right]_0^1 + \left[-(3-x)\frac{1}{3\pi}\cos 3\pi x - \frac{1}{9\pi^2}\sin 3\pi x\right]_1^2 \\ &= -\frac{1}{3\pi}. \end{aligned}$$

(ii). Note that the period of $\tilde{f} = f_{odd}(x)$ is 4, $2015 = 4(504) - 1$, thus

$$\tilde{f}_{av}(2015) = \tilde{f}_{av}(-1) = -\tilde{f}_{av}(1) = -\frac{1+2}{2} = -1.5$$

2. (4 points) Consider the wave equation $16u_{tt} = u_{xx}$, subject to the boundary conditions u(0, t) = u(3, t) = 0 and the initial conditions $u(x,0) = 0, u_t(x,0) = 3\sin(6\pi x) - 2\sin(9\pi x)$. The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right]\sin(\frac{n\pi x}{L}).$$

Find a_n, b_n and the detail solution.

Solution: $c = 1/4,\ L = 3,\ f(x) = 0, g(x) = 3\sin(6\pi x) - 2\sin(9\pi x)$. From

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right]\sin(\frac{n\pi x}{L})$$

we imply that

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi t}{12}) + b_n \sin(\frac{n\pi t}{12})\right]\sin(\frac{n\pi x}{3}).$$

$$u(x,0) = 0, \Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{3}) = 0, \Rightarrow a_n = 0, \quad n \geq 1, \Rightarrow$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi t}{12})\sin(\frac{n\pi x}{3}).$$

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n(\frac{n\pi}{12})\cos(\frac{n\pi t}{12})\sin(\frac{n\pi x}{3}).$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n(\frac{n\pi}{12})\sin(\frac{n\pi x}{3}) = 3\sin(6\pi x) - 2\sin(9\pi x).$$

Comparing coefficients from two sides, when $\frac{n}{3} = 6$, i.e., $n = 18$, $b_{18}(\frac{18\pi}{12}) = 3 \Rightarrow$, $b_{18} = \frac{2}{\pi}$; when $\frac{n}{3} = 9$, i.e., $n = 27$, $b_{27}(\frac{27\pi}{12}) = -2 \Rightarrow$, $b_{27} = -\frac{8}{9\pi}$; All other $b_n = 0$. Hence

$$u(x,t) = \frac{2}{\pi}\sin\left(\frac{3\pi t}{2}\right)\sin(6\pi x) - \frac{8}{9\pi}\sin\left(\frac{9\pi t}{4}\right)\sin(9\pi x)$$

3. (3 points) Solve $u_{xx} = \frac{1}{4}u_t$, $0 < x < 2$, $t > 0$, subject to the boundary conditions $u(0,t) = 0, u(2,t) = 6$, and the initial condition $u(x,0) = 3\sin(3\pi x) - 2\sin\left(\frac{7\pi x}{2}\right) + 3x$.

   (Hint: Let $u(x,t) = v(x) + w(x,t)$ where $v(x)$ is a linear function such that $w(0,t) = w(2,t) = 0$.)

Solution: $\alpha = 2, L = 2$. Let $u(x,t) = v(x) + w(x,t)$ with $v(x) = mx + b$ such that $w(0,t) = w(2,t) = 0$. We have $v(0) = 0, v(2) = 6$. Hence $v(x) = 3x$.
Note that

$$w_{xx} = \frac{1}{4}w_t, \quad w(0,t) = w(2,t) = 0.$$

Thus

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)e^{-(n\pi)^2 t}.$$

Since

$$w(x,0) = u(x,0) - v(x) = 3\sin(3\pi x) - 2\sin\left(\frac{7\pi x}{2}\right) + 3x - 3x = 3\sin(3\pi x) - 2\sin\left(\frac{7\pi x}{2}\right),$$

we imply that

$$w(x,0) = 3\sin(3\pi x) - 2\sin\left(\frac{7\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right),$$

which gives,

$$b_6 = 3, \quad b_7 = -2, \quad b_n = 0 \;\; for \;\; n \neq 6, 7.$$

The final solution is

$$u(x,t) = 3x + 3\sin(3\pi x)e^{-36\pi^2 t} - 2\sin\left(\frac{7\pi x}{2}\right)e^{-49\pi^2 t}.$$